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Multiplication Fact Fluency USING DOUBLES

A sequence of mental math problems using reasoning can boost students' understanding and confidence in performing multiplication.

Do you know a middle school student who does not have solid fluency with multiplication facts and relationships? Many of us do. Not knowing multiplication facts creates a gap in a student's mathematics development and undermines confidence and disposition toward further mathematical learning (Wallace and Gurganus 2005).

When students have a weak grasp on multiplication facts, they often find that many topics are more difficult than they need to be and that some topics are inaccessible. Some of these topics include division, fractions, percents, estimation, and rate and ratio comparisons. Later in secondary school, these students may have difficulty with algebraic factoring, exponential growth, similarity, and trigonometry. And the list goes on.

Indeed, learning multiplication facts is a first step in proportional reasoning, "the capstone of elementary arithmetic and the gateway to higher mathematics" (NRC 2001, p. 242). Proportional reasoning, in turn, is central to success in money management, chemistry, physics, economics, and all phenomena involving change.

We have experienced students' natural interest in doubling, which is a first step in both multiplication and proportional reasoning. We believe that it can be harnessed and used to help students develop multiplication fact fluency (Flowers 1998). Moreover, the process can be done so that students simultaneously use and develop

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number sense and reasoning. We have found that this approach works well with many struggling middle-grades students who have often been advised to "learn your facts" and sent home with a table. Without further guidance, they may end up using flash cards or other tools for memorization.

In contrast, the approach suggested here is not a remedial one of merely drilling, practicing facts, or skip counting. Instead, this approach aims to use and develop problem solving, reasoning, and confidence. Students are asked to identify what they know and to reason their way from using familiar facts to accumulating new ones. The process exploits the idea of *doubling*, a natural and energizing process that many students already have mastered and which helps to initiate a learning strand that leads to proportional reasoning.

Since learning multiplication facts is part of the elementary curriculum, a reasoning-based approach to fact fluency is not typically available in middle school. We would like to fill this gap. We offer these suggestions for use by classroom teachers, special education teachers, parents, and tutors.

A BASIS IN RESEARCH

Our work has many roots. From Gray and Tall (1994), we learn that high-achieving mathematics students are not only quicker but also more strategic in doing mathematics. They see and use opportunities to decompose and recompose numbers. Our work intends to open the door for middle school students who have not yet figured out this powerful key to mathematical thinking.

Our work is also inspired by Confrey (2008) and Confrey and Smith (1995), who have helped us recognize that halving, splitting, and recursion are valuable pathways for instruction. Hiebert and his colleagues (1996) have helped us see that calculation, which is too often presented as a rule-based activity, can instead be conceptualized as a problem to solve. In this light, it becomes amenable to reasoning and problem-solving strategies. From the team of Russell and Economopoulos (2008) we have an example of a full-blown, multiyear curriculum that harnesses and uses these reasoning-based approaches.

This sequence of doubling activities evolved while one author was helping some struggling upper-grade elementary students. She looked at what students knew and analyzed how they might derive what they did not know. It became clear that doubling could play a significant role.

A SEQUENCE TO DEVELOP THE CAPACITY TO DOUBLE

We want students to eventually conceive of doubling as not only adding a number to itself but also a single replicating process that they can do with fluency. Although adults may perceive the tasks to be simple, many students require time to double with ease. **Figure 1** outlines a developmental sequence of mental-math problem sets that allows students to progress from the simplest doubling problems to more complex ones. (An online table presents not only the tasks in each problem set but also suggestions for teachers and illustrates the kind of reasoning that we have seen students develop as they use known relationships to derive new ones.) The time investment, in our experience, is worth it. Strategies for multiplication will come later; first, students should develop strategies for doubling.

This sequence is designed to be used with individual students, but it may also be used with the whole class as a warm-up or filler activity or by students working cooperatively in small groups. For example, groups of students who are still learning to double can ask one another the ques-

Fig. 1 This sequence of mental-math problems allows students to develop competency with doubling.

Problem Set 1
Double digits 5 or less and 10: 1, 2, 3, 4, 5, 10

Problem Set 2
Double digits between 5 and 10: 6, 7, 8, 9

Problem Set 3
Double multiples of 10 to 50: 10, 20, 30, 40, 50

Problem Set 4:
Double small numbers in early decades: 11–15, 21–25, 31–35, 41–45

Problem Set 5:
Double multiples of 10 over 50: 60, 70, 80, 90, 100

Problem Set 6:
Double 5s in later decades: 55, 65, 75, 85, 95

Problem Set 7:
Double large numbers in early decades: 16–19, 26–29, 36–39, 46–49

Problem Set 8:
Double large numbers in later decades: 56–59, 66–69, 76–79, 86–89, 96–99

tions and recycle problems that they are not yet secure in answering. As you study the progression of problems, you will likely see the underlying structure from which it is built.

Each stage is called a *problem set* because we firmly believe that students *are* applying problem-solving strategies to calculation. In most of the sets, students need to apply reasoning strategies that are based on number relationships and the base-ten structure of our place-value system. In so doing, they instinctively recognize the role of decomposing and recomposing numbers, which is characteristic of high achievers in mathematics (Gray and Tall 1994). The reader has also likely recognized the natural way that the distributive property arises: Double 26 is double 20 + double 6. The associative property also arises: Double eight 10s, or $2 \times (8 \times 10)$, is $10 \times$ double 8, or $(2 \times 8) \times 10$.

Although not all the sets are as central as others to developing multiplication fact fluency, we still value students working on them all. There are many benefits, including strengthening place-value understanding, greater mental-math agility, more practice with decomposing and recomposing, and more opportunities to use strategies. We want to take advantage of the opportunity to reinforce an equal-groups meaning of multiplication. To encourage reasoning, provide plenty of wait time.

7 Implementation Suggestions

1. Work on the sets in regular brief sessions rather than too much at once.
2. Mix up the items within each problem set until you are sure that the student can complete the set easily.
3. Include a brief round of doubling at the outset of the lesson and another brief round at the end if you are teaching other topics.
4. Ensure that students are fluent with one problem set before moving on to the next.
5. Begin by doing the sets sequentially.
6. Ensure that students say aloud how they reasoned when completing the more challenging problems. Sharing one's thinking benefits the speaker who clarifies what he has done, benefits the listener who is challenged to follow another's reasoning, and benefits the teacher who can learn the strategies that students are using.
7. Vary the language you use, and connect the language to the equal groups meaning of multiplication. For example, with Problem Set 2 you might say, "What are 2 eights?" "How much is in two groups of size 8?" "What is twice eight?" "What do you get if you double eight?" In particular, we minimize the simple use of "What is two *times* eight?" because it masks the mathematical process of doubling.

FULL-CLASS DOUBLING

Another activity that helps build fluency in doubling and when working on multiplication facts is "doubling around the room." The first student says "1," and each consecutive student doubles the previous number. The students work mentally. The teacher records the number sequence on the overhead projector or the board. The written record helps students to do the mental doubling. Use a cooperative, not competitive, approach: Allow a student on either side of the one whose turn it is to help. To keep the task manageable, you might ask, "How far can we go as a class if we double, starting with 1?" Again, for the more difficult problems, ask students to say what strategy they used. Although the numbers get large quickly (one of the features that teachers may want to discuss), the series is easily generated by students who have been working on doubling strategies. Watch, listen, and restart if necessary. In time, producing the list below mentally is attainable by middle-grades classes:

1; 2; 4; 8; 16; 32; 64; 128;
256; 512; 1024; 2048; 4096;
8192; 16,384; 32,768

Usually, students' strategies converge to resemble one of the following:

- 2×256: Twice 2 hundred is 4 hundred and twice 50 is 100, making 5 hundred. Then twice 6 is 12, making 512.
- 2×256: Twice 25 tens is 50 tens, or 500. Then twice 6 is 12, making 512.

One colleague who has used this activity with sixth graders in the Detroit Public Schools commented, "It's been amazing to me to see the mental math strategies being used. Also, I have seen how it is accessible to kids on many different ability levels" (Ahmed, personal communication, 2009).

CONNECTING TO MULTIPLICATION

Once students are familiar with the doubling process, work on connecting the sequence of doubling tasks to multiplication fact fluency. Using several even factors, such as 2s, 4s, and 8s, can be natural starting points for doubles. Once the 3s are known, the 6s and the 12s can be placed in the mix. But doubling can still play a role with odd factors. For example, the 3s can be seen as the sum of 2s and 1s. (More detail on 3s follows.)

Students are usually familiar with the 5s from earlier work counting by 5s or halving 10s. For other factors, students commonly decompose one factor, multiply the parts by the other factor, then add (or subtract) the results, which is the distributive property. For example, the facts for 9s are 10s minus 1s: nine 8s are ten 8s minus one 8, or 80 – 8 = 72. This leaves the 7s as the one family of facts to be learned. But with commutativity and the other facts in place, 7 × 7 may be the only single fact that students need to memorize.

THE FACTS ABOUT 3

Despite the fact that 3 is a small number, some students need to work on learning the 3s. We believe that this activity can be done through reasoning and problem solving so that students learn *more* than the 3s facts: They also develop general strategies for other factors.

To learn the 3s, we focus on developing strategies for calculating addition pairs and recognizing that multiplication can be thought of as repeated addition. With this foundation, a student can think of 3 × 7 as meaning three 7s (7 + 7 + 7). Students can use the doubling ideas already practiced and can be prompted in this way: "What is two groups of 7?" "How can you build from there?" In so doing, they can see that two groups of 7 is 14, so three groups of 7 is 14 + 7. The latter may be computed by decomposing the 7 as (6 + 1) to find

$$\begin{aligned} 14 + 7 &= 14 + (6 + 1) \\ &= (14 + 6) + 1 \\ &= 20 + 1. \end{aligned}$$

(This strategy for addition would have come up earlier in the doubling work.)

Take time with your students to help them figure out the facts for 3s and to practice them for fluency.

Fig. 2 Using a table with columns "facts I know" and "facts I need to learn" gives students and their teacher information about both easy and difficult products.

Facts I Know	Facts I Need to Learn
2 × 9	8 × 7
3 × 5	6 × 4
10 × 7	3 × 7
6 × 1	7 × 7

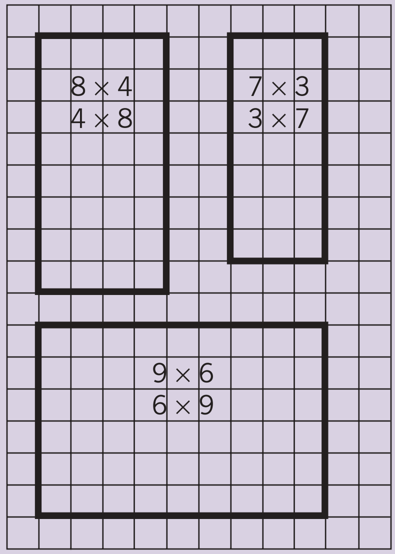
INVENTORY OF KNOWN FACTS

At some point in the process of helping a student solidify multiplication facts, do an inventory to sort the facts that are easily known from those that are more difficult to produce. Ask the student for products for different factor pairs (not in any particular order) from a table or other source. (Check off each fact to ensure that you do a full inventory.) As you state the factor pairs, see whether the student names the product easily or not. Use a record sheet (see **fig. 2**), and let the student use the columns to record "facts I know" or "facts I need to learn." The list of known facts is often reassuringly long, containing 1s and 2s and some 3s, 5s, 9s, and 10s.

ARRAYED FOR ALL TO SEE

At this point in the process, students can use arrays. Ask students to sketch rectangles on centimeter or quarter-inch grid paper of the appropriate dimensions for the "facts they need to learn" (see **fig. 3**). These images help students visualize properties. For example, the 3 × 7 rectangle can show 3 groups (columns) of 7 as well as 7 groups (rows) of 3, thereby illustrating the commutativity of multiplication. The array also supports a view of the distributive property. For example, 7 groups of 3 can be seen as 5 groups of 3 plus 2 groups of 3 by darkening

Fig. 3 Using an array model can help students learn the products that they have yet to master.



an appropriate segment on the grid. In addition, the image connects an equal-groups meaning of multiplication with an area meaning.

As the teacher points to different arrays, the student can be asked to find the product. For selected pairs of numbers, ask a student to explain his or her thinking. Once a fact becomes solidly understood, the array may be removed from the set. When a student is stuck, say, for 8 × 7, the teacher may ask,

- "What fact do you know that you can use to find eight 7s?"
- "Does knowing four 7s help you?"
- "Does knowing two 7s help?"

The goal is for the student to mentally ask these sorts of questions, without prompting, and reason that double 7 is 14, double 14 (four 7s) is 28, and, finally, double 28 (eight 7s) is 56. Doubling questions such as these, focused on mathematical relationships, lead to the reasoning that stronger students often develop independently. Other students benefit when this kind of reasoning is revealed explicitly.

Although individual differences are seen in how students respond to this sequence of activities, in our experience there has always been progress. More important, the focus shifts from simply learning facts to building relationships to derive products. We have found that students grow not only in knowledge but also in mathematical reasoning and confidence. We want them to learn not simply the products for factor pairs but the reasoning they can use to construct these relationships for themselves.

LOOKING BACK AND LOOKING AHEAD

The thinking required for these activities connects to the five Process Standards (NCTM 2000), which are Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. Consider, for example, the fact 8 × 7, which we know is learned late by many students. There are multiple solution paths a student might use. One method involves repeated doubling. The student might also decompose 8 as 5 + 3 and find the sum of five 7s and three 7s: 35 + 21 = 56. Another way might be to subtract two 7s from ten 7s, to get 70 – 14 = 56 for eight 7s.

What processes do we find in this example? For one, the student is *problem solving*. In the terminology of Hiebert and his colleagues (1996), we are allowing the work of calculation to be "problematic" (a problem to be solved). Clearly, there is *reasoning* and the seeds of *proof*. The student is using careful reasoning that will lead to definite, reliable conclusions (Hersh 2009) and applying logic to previously known or accepted meanings, facts, and relationships (e.g., meanings for multiplication, already established number facts) to generate new relationships. When we ask the student to verbalize his or her thinking, *communication* occurs. When we ask that symbols or diagrams be used, he or she is employing *representation*. The work also makes clear *connections*; for example, 8 is double 4, which in turn is double 2. The distributive property (with or without being named) is a tool for building multiplication facts.

When we think about this sequence of doubling activities in the larger scheme of students' previous and later learning, we see that it provides valuable links both to work that comes before and work that comes later. In terms of previous learning, these activities strengthen and continue to build on an understanding of useful mathematical concepts, such as the place-value structure of our base-ten numeration system, the distributive property, and decomposing and recomposing. For future learning, these activities pave the way for such important work ahead as multiplicative comparisons, rational numbers, probability, exponential growth, and proportional reasoning. It should also be noted that these activities are significant confidence- and motivation-boosters for many students.

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A detailed table of problem sets with prompts and examples are appended to the online version of this article at **www.nctm.org/mtms.**

Multiplication Fact Fluency Using Doubles

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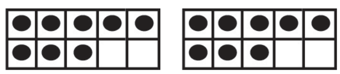
Problem Set	Tasks for Students	Prompts and Suggestions for Teachers or Tutors	Student Reasoning That May Arise
1	Double digits 5 or less and 10: 1, 2, 3, 4, 5, 10	*"Tell me, what is 'double 3'?" "What is in two groups of 4?" "How much is twice 5?"* Here at the outset and throughout, minimize use of the word *times.* Help the student think in terms of two groups of a given size. Mix the facts in each set. In particular, for a fact where the student is hesitant, return to it often.	Students use addition facts for doubles.
2	Double digits between 5 and 10: 6, 7, 8, 9	*"How do you think about double 6?" "What is your strategy for figuring out twice 8?"* Introduce, if needed, tens frames to support student visualization. Here is 8 + 8. 	Students decompose using 5 as a benchmark; they double 5 and the additional amount. Example: "Double 8 is double (5 + 3) or 10 + 6."
3	Double multiples of 10 to 50: 10, 20, 30, 40, 50	"*How much is twice 30?" "What does 30 mean?" "How do you read* [*point to the symbol '30'*] *to say what it means?"* Use language to help students focus on place-value meanings, asking, for example, what does 30 mean? [3 tens] If the student cannot say "three tens," then inform him or her and ask comparable questions as you continue.	Students use place-value language to explain. Example: "Double 30 is double three 10s, or six 10s, or 60."
4	Double small numbers in early decades: 11–15, 21–25, 31–35, 41–45	*"Double 42." "What strategy can you use to figure it out?"* *"What does 42 mean?" "How does thinking about what it means help you double it?"* If students do not do so themselves, encourage them to focus first on the 10s. The idea is to stay focused on the meaning of the numbers, not to replicate the paper algorithm.	Students decompose and recompose. Examples: "Double 43 is double 40 plus double 3, or 80 + 6 = 86." "Double 45 is double 40 plus double 5, or 80 plus 10, or 90."

Problem Set	Tasks for Students	Prompts and Suggestions for Teachers or Tutors	Student Reasoning That May Arise
5	Double multiples of 10 over 50: 60, 70, 80, 90, 100	*What is twice 90?" "What is the meaning of 90?" "What do you know that you can use?"* Again, focus on place-value language: 90 is nine 10s. Students may need support in thinking about what eighteen 10s means. "*What is eighteen 10s the same as?" "If you count by 10s eighteen times, where do you land?"* Recognizing the equivalence of eighteen 10s and 180 is important.	Students use place-value language to explain. Examples: "Double 90 is double nine 10s, or eighteen 10s, or 180." "Ten 10s is 100, so 8 more 10s make 180."
6	Double 5s in later decades: 55, 65, 75, 85, 95	*"How can you figure out twice 95?"* *"What does the number mean?"* *"How can you use the meaning?"* Again, encourage students to focus first on the 10s.	Students decompose and recompose. Examples: "Double 95 is double 90 plus double 5, or 180 + 10 = 190."
7	Double large numbers in early decades: 16–19, 26–29, 36–39, 46–49	*"What is double 28? How can you use something you already determined to help you figure this out?"*	Students decompose and recompose, mainly by 10s and 1s. Example: "Double 28 is double 20 plus double 8, or 40 plus 16. That is 40 + 10 + 6, or 56."
8	Double large numbers in later decades: 56–59, 66–69, 76–79, 86–89, 96–99	Continue as above, focusing on meanings of numbers and deriving new facts from known facts.	Students decompose and recompose, mainly by 10s and 1s. Some strategies may involve subtraction from a larger multiple of 10. Examples: "Double 68 is double 60 + double 8, or 120 + 16. That is 120 + 10 + 6, or 136." "Double 68 is double 70 − double 2, or 140 – 4 = 136."